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A generalization of the Mitchell Lemma: The Ulmer Theorem and the Gabriel–Popescu Theorem revisited

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A R T I C L E I N F O

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A B S T R A C T

We prove a generalization of the Mitchell Lemma, and we show that it is a key lemma that can be used in order to deduce in a unified easier way several important results. Thus, the Ulmer Theorem, the generalized Gabriel–Popescu Theorem and the generalized Takeuchi Lemma are all consequences of the generalized Mitchell Lemma.

1. Introduction

The appearance of the Gabriel–Popescu Theorem [5] in 1964 has produced a lot of excitement in the mathematical literature. The theorem shows that each Grothendieck category is equivalent to a quotient category of a module category. More precisely, if $\mathcal{A}$ is a Grothendieck category with a generator G, $R = \mathrm{End}_{\mathcal{A}}(G)$ is the endomorphism ring of G and $\mathrm{Mod}(R)$ is the category of unitary right R-modules, then the functor $T : \mathcal{A} \to \mathrm{Mod}(R)$ defined by $T(X) = \mathrm{Hom}_{\mathcal{A}}(G, X)$ on objects X of $\mathcal{A}$ and by $T(f) = \mathrm{Hom}_{\mathcal{A}}(G, f) : T(X) \to T(Y)$ on morphisms $f : X \to Y$ in $\mathcal{A}$ is fully faithful and has an exact left adjoint S [5]. The original proof seemed rather complicated, especially the part on the exactness of the functor S. Afterwards, it was revisited by several authors; among the most elegant and short proofs, in chronological order, are those by Takeuchi [15], Ulmer [16] and Mitchell [11]. The latter uses an ingenious lemma [11, Lemma], referred to as the Mitchell Lemma, and a different point of view which employs the existence of enough injective objects in any Grothendieck category. Our goal is to extend this lemma from module categories to functor categories, and to show how it can be used in order to obtain in a unified easier way the Ulmer Theorem on the exactness of S, a generalized Gabriel–Popescu Theorem and a generalized Takeuchi Lemma. Having the generalized Mitchell Lemma, the order in which we prove the results will be different than usual: first, we show that the functor T remains full and faithful on a certain subclass of objects of $\mathcal{A}$, then we give a new proof of the Ulmer Theorem on the exactness of S, and finally we deduce a generalized Gabriel–Popescu Theorem and a generalized Takeuchi Lemma. The key ingredients will be the existence of enough injective objects in Grothendieck categories [7] and the Baer criterion for injectivity in Grothendieck categories with a generating family [1].

2. A generalization of the Mitchell Lemma

We shall use the following setting and notation throughout the paper. For general terminology for categories we refer the reader to [10]. Let $\mathcal{A}$ be a Grothendieck category, that is, an AB5 category with a generator. Let $\mathcal{U}$ be a set of objects of $\mathcal{A}$.

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We denote by $\mathrm{Gen}(\mathcal{U})$ the full subcategory of $\mathcal{A}$ consisting of the $\mathcal{U}$-generated objects, that is, objects A of $\mathcal{A}$ for which there is an epimorphism from a coproduct of objects of $\mathcal{U}$ to A. Then in general $\mathrm{Gen}(\mathcal{U})$ is a preabelian subcategory of $\mathcal{A}$ (i.e., it has kernels and cokernels). In fact, the kernel of a morphism f in $\mathrm{Gen}(\mathcal{U})$ is the trace of $\mathcal{U}$ in the kernel $\mathrm{Ker}(f)$ of f in $\mathcal{A}$ (i.e., the sum of all images of morphisms $h : U \to \mathrm{Ker}(f)$ with $U \in \mathcal{U}$), whereas the cokernel of f in $\mathrm{Gen}(\mathcal{U})$ is the same as its cokernel in $\mathcal{A}$. For instance, if $\mathcal{U}$ consists only of the group $\mathbb{Q}$ of rational numbers, then $\mathbb{Q}/\mathbb{Z} \in \mathrm{Gen}(\mathbb{Q})$, and the canonical morphism $f : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism in the category $\mathrm{Gen}(\mathbb{Q})$. We denote by $\mathcal{A}_{\mathcal{U}}$ the class of objects M of $\mathcal{A}$ such that for every morphism $f : \bigoplus_{U\in F} U \to M$ in $\mathcal{A}$ with F a finite subset of $\mathcal{U}$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.

Let $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ be the category whose objects are the additive contravariant functors from $\mathcal{U}$ to the category Ab of abelian groups, and whose morphisms are the natural transformations between such functors. It is well-known that $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is a Grothendieck category, and the contravariant representable functors $(h_U)_{U\in\mathcal{U}}$, where $h_U = \mathrm{Hom}_{\mathcal{A}}(-, U)$, form a generating family of finitely generated projective objects for $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$.

Consider the functor $T : \mathcal{A} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ defined by $T(X) = \mathrm{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{U}}$ on objects A of $\mathcal{A}$, and by $T(f) = \mathrm{Hom}_{\mathcal{A}}(-, f)|_{\mathcal{U}} : T(X) \to T(Y)$ on morphisms $f : X \to Y$ between objects in $\mathcal{A}$. Then it is known that T has a left adjoint $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ (e.g., see [4, Chapitre 5, Proposition 1.1], [16], [3, p. 84] and [10, Chapter IV, Theorem 5.2]). In particular, we have $S(h_U) = U$ for every $U \in \mathcal{U}$. We denote by $\nu : ST \to 1_{\mathcal{A}}$ and $\eta : 1_{(\mathcal{U}^{\mathrm{op}},\mathrm{Ab})} \to TS$ the counit and the unit of the adjunction (S, T) respectively.

For a morphism $f : A \to B$ in $\mathcal{A}$, we denote its kernel by $\ker(f) : \mathrm{Ker}(f) \to A$ and its image by $\mathrm{im}(f) : A \to \mathrm{Im}(f)$. Now we are ready to establish a generalization of the Mitchell Lemma [11, Lemma].

Lemma 2.1. *Let $\mathcal{U}$ be a set of objects of $\mathcal{A}$, A and B objects of $\mathcal{A}$ with $A \in \mathcal{A}_{\mathcal{U}}$, M_A a subobject of $T(A)$ and $G : M_A \to T(B)$ a morphism in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$. Define $M = \bigcup_{U\in\mathcal{U}} M_A(U)$ and for every $m \in M$ take some $U_m = U \in \mathcal{U}$ for which $m \in M_A(U)$. For every $m \in M$, denote by $u_m : U_m \to \bigoplus_{m\in M} U_m$ the canonical injection. Let $\psi : \bigoplus_{m\in M} U_m \to A$ be the unique morphism such that $\psi u_m = m$ for every $m \in M$, and $\phi : \bigoplus_{m\in M} U_m \to B$ the unique morphism such that $\phi u_m = G_U(m)$ for every $m \in M_A(U)$. Then ϕ factors through $\mathrm{Im}(\psi)$.*

Proof. The existence and uniqueness of the morphisms ψ and ϕ follow by the universal property of the coproduct. Define $\mu = \ker(\psi) : K \to \bigoplus_{m\in M} U_m$. The required assertion is equivalent to $\phi\mu = 0$. Let F be a finite subset of M, and consider the morphism $\beta = \sum_{m\in F} u_m p_m$, where $p_m : \bigoplus_{m\in F} U_m \to U_m$ is the mth canonical projection. By taking the pullback of μ and β, we have the induced commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "\mu'"] \arrow[d, "\alpha"] & \bigoplus_{m\in F} U_m \arrow[r, "\pi'"] \arrow[d, "\beta"] & \mathrm{Coker}(\mu') \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & K \arrow[r, "\mu"] & \bigoplus_{m\in M} U_m \arrow[r, "\pi"] \arrow[d, "\phi"] \arrow[dr, "\psi"] & \mathrm{Coker}(\mu) \arrow[r] \arrow[d, "i"] & 0 \\
 & & B & A &
\end{tikzcd}$$

where $i : \mathrm{Coker}(\mu) \to A$ is the inclusion morphism. By [14, Chapter IV, Proposition 5.1], $\mu' = \ker(\pi\beta)$; hence $\mu' = \ker(i\pi\beta) = \ker(i\gamma\pi')$. Since $A \in \mathcal{A}_{\mathcal{U}}$ and $K' = \mathrm{Ker}(i\gamma\pi')$, we have $K' \in \mathrm{Gen}(\mathcal{U})$. We continue the proof as in the Mitchell Lemma [11, Lemma], giving only some details. Since $\mathcal{A}$ is an AB5 category, it follows that $\phi\mu = 0$ if and only if $\phi\mu\alpha = 0$ for every finite subset F of $\mathcal{U}$ if and only if $\phi\mu\alpha\delta = 0$ for every finite subset F of $\mathcal{U}$ and every morphism $\delta : V \to K'$ with $V \in \mathcal{U}$. Now let $\delta : V \to K'$ be a morphism with $V \in \mathcal{U}$ and let $F = \{m_1, \ldots, m_n\}$, where each $m_i \in M_A(U_i)$. Since G is a functorial morphism, we have the following commutative diagram:

$$\begin{tikzcd}
M_A(U_i) \arrow[r, "M_A(p_{m_i}\mu'\delta)"] \arrow[d, "G_{U_i}"] & M_A(V) \arrow[d, "G_V"] \\
\mathrm{Hom}_{\mathcal{A}}(U_i, A) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(p_{m_i}\mu'\delta, 1_A)"'] & \mathrm{Hom}_{\mathcal{A}}(V, A)
\end{tikzcd}$$

Then it follows that:

$$\begin{aligned}
\phi\mu\alpha\delta &= \phi\left(\sum_{i=1}^{n} u_{m_i} p_{m_i}\right)\mu'\delta = \sum_{i=1}^{n} G_{U_i}(m_i) p_{m_i}\mu'\delta = \sum_{i=1}^{n} G_V(m_i p_{m_i}\mu'\delta) \\
&= G_V\left(\sum_{i=1}^{n} m_i p_{m_i}\mu'\delta\right) = G_V(\psi\mu\alpha\delta) = 0,
\end{aligned}$$

as needed. $\square$

3. Applications

In this section we show how the generalized Mitchell Lemma (Lemma 2.1) can be used in order to deduce several important results, such as the Ulmer Theorem, the generalized Gabriel–Popescu Theorem and the generalized Takeuchi Lemma. But first we establish the following proposition.

Proposition 3.1. *Let A and B be objects of $\mathcal{A}$ with $A \in \mathcal{A}_{\mathcal{U}} \cap \mathrm{Gen}(\mathcal{U})$. Then:*

(i) *The map $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{(\mathcal{U}^{\mathrm{op}},\mathrm{Ab})}(T(A), T(B))$ defined by the assignment $f \mapsto T(f)$ is bijective.*
(ii) *The canonical morphism $\nu_A : ST(A) \to A$ is an isomorphism.*

Proof. (i) Let $f : A \to B$ be a morphism such that $T(f) = 0$. Note that $T(f)(U) : \mathrm{Hom}_{\mathcal{A}}(U, A) \to \mathrm{Hom}_{\mathcal{A}}(U, B)$ is defined by $T(f)(U)(\alpha) = f\alpha$ for every $\alpha \in \mathrm{Hom}_{\mathcal{A}}(U, A)$. Then $T(f)(U) = 0$ for every $U \in \mathcal{U}$, and furthermore, $T(f)(U)(\alpha) = f\alpha = 0$ for every $\alpha \in \mathrm{Hom}_{\mathcal{A}}(U, A)$. Since $A \in \mathrm{Gen}(\mathcal{U})$, we have $f = 0$.

Now let $G : T(A) \to T(B)$ be a morphism. We use the notation from Lemma 2.1. Since $A \in \mathcal{A}_{\mathcal{U}}$, by using Lemma 2.1 for $M_A = T(A)$ there is a morphism $f : A \to B$ in $\mathcal{A}$ such that $f\psi = \phi$. Then for every $m \in \mathrm{Hom}_{\mathcal{A}}(U, A) = T(A) = M_A$ and every $U \in \mathcal{U}$ we have $T(f)(U)(m) = fm = f\psi u_m = \phi u_m = G_U(m)$. It follows that $T(f) = G$.

(ii) By (i) and the adjunction we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(A, B) \arrow[r, "u\mapsto T(u)", "\cong"'] \arrow[d, "u\mapsto u\nu_A"'] & \mathrm{Hom}_{(\mathcal{U}^{\mathrm{op}},\mathrm{Ab})}(T(A), T(B)) \arrow[d, equal] \\
\mathrm{Hom}_{\mathcal{A}}(ST(A), B) \arrow[r, "\cong", "v\mapsto T(v)\eta_{T(A)}"'] & \mathrm{Hom}_{(\mathcal{U}^{\mathrm{op}},\mathrm{Ab})}(T(A), T(B))
\end{tikzcd}$$

Indeed, for every $u \in \mathrm{Hom}(A, B)$, we have $T(u)T(\nu_A)\eta_{T(A)} = T(u)$. It follows that the morphism $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(ST(A), B)$ given by $u \mapsto u\nu_A$ is a functorial isomorphism in B, and so the covariant functors $h^A = \mathrm{Hom}_{\mathcal{A}}(A, -) \cong \mathrm{Hom}_{\mathcal{A}}(ST(A), -) = h^{ST(A)}$ are isomorphic via the morphism ν_A. Now it follows from the Yoneda Lemma that ν_A is an isomorphism. $\square$

We shall need the following Baer-type criterion for injectivity (see [1, Proposition 1.13, p. 11]). Recall that an object A of a Grothendieck $\mathcal{A}$ is called V-*injective* for some object V of $\mathcal{A}$ if for every subobject X of V, every morphism $X \to A$ extends to a morphism $V \to A$.

Proposition 3.2. *Let $\mathcal{A}$ be a Grothendieck category with a generating family $\mathcal{V}$. Then an object A of $\mathcal{A}$ is injective if and only if it is V-injective for every $V \in \mathcal{V}$.*

Now we are in a position to give a new proof of the Ulmer Theorem [16, Theorem] essentially based on the generalized Mitchell Lemma and the above Baer-type criterion for injectivity.

Theorem 3.3 (*Ulmer*)**.** *The functor $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ is exact if and only if $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$.*

Proof. Note that since $\mathcal{A}$ has enough injectives, S is exact if and only if T preserves injective objects.

Assume that $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$ and let Q be an injective object of $\mathcal{A}$. In order to prove that $T(Q)$ is injective in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ it is enough to check the Baer-type criterion given by Proposition 3.2, that is, injectivity relative to any generator $h_U = T(U)$ with $U \in \mathcal{U}$. To this end, let $U \in \mathcal{U}$, and let $j : M_U \to T(U)$ be a monomorphism and $f : M_U \to T(Q)$ a morphism in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$. Define $M = \bigcup_{V\in\mathcal{U}} M_U(V)$ and for every $m \in M$ take some $V_m = V \in \mathcal{U}$ for which $m \in M_U(V)$. Now use Lemma 2.1 for $A = U$ and $B = Q$ to deduce the existence of a factorization of the morphism $\phi : \bigoplus_{m\in M} V_m \to Q$ through $\mathrm{Im}(\psi)$ by some morphism $h : \mathrm{Im}(f) \to Q$. By the injectivity of Q, h extends to a morphism $u : U \to Q$, and we have $u\psi = \phi$. It follows that $T(u)j = f$, and so $T(Q)$ is injective. Hence S is exact.

Conversely, assume that S is exact. Let $U \in \mathcal{U}$ and let $f : U_1 \oplus \cdots \oplus U_n \to U$ be a morphism in $\mathcal{A}$ for some objects $U_1, \ldots, U_n \in \mathcal{U}$. Since T is left exact, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & ST(\mathrm{Ker}(f)) \arrow[r] \arrow[d, "\nu_{\mathrm{Ker}(f)}"'] & ST(U_1 \oplus \cdots \oplus U_n) \arrow[r] \arrow[d, "\nu_{U_1\oplus\cdots\oplus U_n}"] & ST(U) \arrow[d, "\nu_U"] \\
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] & U_1 \oplus \cdots \oplus U_n \arrow[r] & U
\end{tikzcd}$$

Since we have $ST(U) = U$ and ST commutes with finite direct sums, it follows that ν_U and $\nu_{U_1\oplus\cdots\oplus U_n}$ are isomorphisms. Then $\nu_{\mathrm{Ker}(f)}$ is an isomorphism. By the construction of S, we have $S(T(\mathrm{Ker}(f))) \in \mathrm{Gen}(\mathcal{U})$, whence $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$. $\square$

As a consequence now we may immediately deduce the following generalized version of the classical Gabriel–Popescu Theorem [5].

Theorem 3.4 (*Generalized Gabriel–Popescu*)**.** *Assume that $\mathcal{U}$ is a family of generators of $\mathcal{A}$. Then $T : \mathcal{A} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is a full and faithful functor, and its left adjoint $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ is exact.*

Proof. It follows by Proposition 3.1, Theorem 3.3 and the fact that $\mathcal{A} = \mathrm{Gen}(\mathcal{U})$. $\square$

Remark 3.5. (i) Note that Theorem 3.4 shows in fact that $\mathcal{A}$ is a quotient category of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$, and the classical Gabriel–Popescu Theorem is obtained for $\mathcal{U} = \{U\}$.

(ii) The generalized Gabriel–Popescu Theorem 3.4 was previously established by some authors (see [2,6,12,13]), but their proofs mainly use the rather complicated approach of the original Gabriel–Popescu Theorem.

We have a characterization of the situation when the above functor T is an equivalence of categories. We omit the proof since the result can be easily deduced from Theorem 3.4, and has previously appeared in various forms (e.g., see [10] or [9]).

Corollary 3.6. *Assume that $\mathcal{U}$ is a family of generators of $\mathcal{A}$. Then $T : \mathcal{A} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is an equivalence of categories if and only if $\mathcal{U}$ is a family of small projective objects of $\mathcal{A}$.*

We may refine the above equivalence as follows. Recall that $\mathrm{Ker}(S)$ is the class of objects K in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ such that $S(K) = 0$, and an object M of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is called $\mathrm{Ker}(S)$-*closed* if for every morphism $g : L \to N$ in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ with $\mathrm{Ker}(g), \mathrm{Coker}(g) \in \mathrm{Ker}(S)$, every morphism $L \to M$ extends to a morphism $N \to M$.

Proposition 3.7. *Assume that $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$. Then there is an equivalence of categories between $\mathcal{A}_{\mathcal{U}} \cap \mathrm{Gen}(\mathcal{U})$ and the full subcategory $\mathcal{B}$ of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ consisting of the $\mathrm{Ker}(S)$-closed objects X such that $S(X) \in \mathcal{A}_{\mathcal{U}} \cap \mathrm{Gen}(\mathcal{U})$.*

Proof. Note that, since $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$, S is exact by Theorem 3.3.

First, let $A \in \mathcal{A}_{\mathcal{U}} \cap \mathrm{Gen}(\mathcal{U})$. In order to show that $T(A)$ is $\mathrm{Ker}(S)$-closed, let $g : X \to X'$ and $f : X \to T(A)$ be morphisms in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ such that $\mathrm{Ker}(g), \mathrm{Coker}(g) \in \mathrm{Ker}(S)$. Then $S(g) : S(X) \to S(X')$ is an isomorphism, and so there is a morphism $\gamma : S(X') \to A$ such that $\gamma S(g) = \nu_A S(f)$. By naturality it follows that

$$\begin{aligned} T(\gamma)\eta_{X'}g &= T(\gamma)TS(g)\eta_X = T(\gamma S(g))\eta_X = T(\nu_A S(f))\eta_X \\ &= T(\nu_A)TS(f)\eta_X = T(\nu_A)\eta_{T(A)}f = f, \end{aligned}$$

which shows that $f : X \to T(A)$ extends to $T(\gamma)\eta_{X'} : X' \to T(A)$. The uniqueness of such an extension follows by the adjunction. Thus $T(A)$ is $\mathrm{Ker}(S)$-closed. By Proposition 3.1, we have $ST(A) \cong A$. It is then clear that $T(A) \in \mathcal{B}$.

Now let $X \in \mathcal{B}$. Then $S(X) \in \mathcal{A}_{\mathcal{U}} \cap \mathrm{Gen}(\mathcal{U})$. We claim that $\eta_X : X \to TS(X)$ is an isomorphism, which will end the proof. Since $\nu_{S(X)}S(\eta_X) = 1_{S(X)}$ and $\nu_{S(X)}$ is an isomorphism by Proposition 3.1, so is $S(\eta_X)$. Hence $\mathrm{Ker}(\eta_X), \mathrm{Coker}(\eta_X) \in \mathrm{Ker}(S)$. Since X is $\mathrm{Ker}(S)$-closed, there is a morphism $\varepsilon : TS(X) \to X$ such that $\varepsilon\eta_X = 1_X$, and so $TS(X) \cong \mathrm{Im}(\eta_X) \oplus Y$ for some object Y. But Y is $\mathrm{Ker}(S)$-closed and $Y \in \mathrm{Ker}(S)$, and so $Y = 0$ and $TS(X) \cong \mathrm{Im}(\eta_X)$. It follows that η_X is an isomorphism. $\square$

For his short proof of the classical Gabriel–Popescu Theorem [15], Takeuchi proved a fundamental lemma, which may also be of independent interest. Now we can immediately obtain a generalization of this Takeuchi Lemma (we point out that the original proof by Takeuchi did not use the result that every Grothendieck category has enough injective objects).

Corollary 3.8. *Let A be an object of $\mathcal{A}$, let Y_A be a subobject of $T(A)$ and denote by $i : Y_A \to T(A)$ the inclusion morphism.*

(i) *If $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$, then $S(i)$ is a monomorphism.*
(ii) *If $A \in \mathcal{A}_{\mathcal{U}} \cap \mathrm{Gen}(\mathcal{U})$, then the canonical morphism $\nu_A : ST(A) \to A$ is an isomorphism.*

Proof. (i) Since $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$, S is exact by Theorem 3.3. Then $S(i)$ is a monomorphism.
(ii) This follows by Proposition 3.1. $\square$

Remark 3.9. One can extend all the above results by considering a functor $F : \mathcal{V} \to \mathcal{A}$ from a small preadditive category $\mathcal{V}$ to $\mathcal{A}$ instead of the inclusion functor $F : \mathcal{U} \to \mathcal{A}$. This approach was used by Lowen [8] in order to obtain a generalization of the Gabriel–Popescu Theorem.

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References

[1] T. Albu, C. Năstăsescu, Relative Finiteness in Module Theory, Marcel Dekker, 1984.
[2] F. Castaño-Iglesias, P. Enache, C. Năstăsescu, B. Torrecillas, Gabriel–Popescu type theorems and applications, Bull. Sci. Math. 128 (2004) 323–332.
[3] P. Freyd, Abelian Categories, Harper and Row, New York, 1964.
[4] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962) 323–448.
[5] P. Gabriel, N. Popescu, Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes, C. R. Acad. Sci. Paris 258 (1964) 4188–4190.
[6] G. Garkusha, Grothendieck categories, Algebra i Analiz 13 (2001) 1–68 (Russian). Engl. transl. in St. Petersburg Math. J. 13 (2002) 149–200.
[7] A. Grothendieck, Sur quelques points d'algèbre homologique, I, Tohoku Math. J. (2) 9 (1957) 119–221.
[8] W. Lowen, A generalization of the Gabriel–Popescu theorem, J. Pure Appl. Algebra 190 (2004) 197–211.
[9] C. Menini, Gabriel–Popescu type theorems and graded modules, in: Perspectives in Ring Theory, Antwerp, 1987, Kluwer, Dordrecht, 1988, pp. 239–251.
[10] B. Mitchell, Theory of Categories, Academic Press, New York, 1965.
[11] B. Mitchell, A quick proof of the Gabriel–Popesco theorem, J. Pure Appl. Algebra 20 (1981) 313–315.
[12] C. Năstăsescu, C. Chiteş, A version of the Gabriel–Popescu theorem, An. Şt. Univ. Ovidius Constanţa 18 (2010) 189–200.
[13] M. Prest, Elementary torsion theories and locally finitely presented categories, J. Pure Appl. Algebra 18 (1980) 205–212.
[14] B. Stenström, Rings of quotients, in: Grundleheren der Math., 217, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
[15] M. Takeuchi, A simple proof of Gabriel and Popesco's theorem, J. Algebra 18 (1971) 112–113.
[16] F. Ulmer, A flatness criterion in Grothendieck categories, Invent. Math. 19 (1973) 331–336.